A Common Trick of Gödel and Tarski

Saeed Salehi

World Logic Day — 14 January 2021

Iranian Association for Logic

World Logic Day

- World Philosophy Day (since 2002):
  The Third Thursday of November.

- World Mathematics Day (since 2019 by UNESCO):
  March 14th: $3.14 \approx \pi$. — The former Pi Day (since 1988).

- World Logic Day (since 2019):
  14th of January — Gödel's death (1978) & Tarski's birth (1901).

What is Common in the *proofs* of Gödel and Tarski?

Gödel (1931): The (First) Incompleteness Theorem (Semantically):

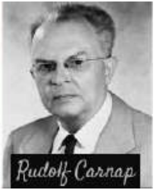
Every *sound* and *recursively enumerable* theory is <u>in</u>complete.

Tarski (1933): The Undefinability Theorem (Semantically):

Arithmetical `truth` is not arithmetically definable.

Carnap (1934): The Diagonal Lemma (Semantically):

For each $\Psi(x)$ there is a sentence η such that $\mathbb{N} \models \eta \equiv \Psi(\#\eta)$.



The *Hocus Pocus* Proof of the Diagonal Lemma

(2002) MCGEE, VANN; The First Incompleteness Theorem, *Handouts of the Course "Logic II"*. https://bit.ly/301QLTA

> "I don't know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma works. It has a rabbit-out-of-a-hat quality for everyone."

(2006) GAIFMAN, HAIM; Naming and Diagonalization, from Cantor to Gödel to Kleene, *Logic Journal of the IGPL* 14(5):709–728.

> "The brevity of the proof [of the Diagonal Lemma] does not make for transparency; it has the aura of a magician's trick."

(2008) WASSERMAN, WAYNE URBAN; It ***Is*** "Pulling a Rabbit Out of the Hat": Typical Diagonal Lemma "Proofs" Beg the Question, *Social Science Research Network*, 1–11. DOI: 10.2139/ssrn.1129038

Abracadabra & … The Diagonal-Free Proofs

(2004) Kotlarski, Henryk; The Incompleteness Theorems After 70 Years, *Annals of Pure and Applied Logic* 126(1-3):125–138.

> "being very intuitive in the natural language, is highly unintuitive in formal theories like Peano arithmetic. In fact, the usual proof of the diagonal lemma … is short, but tricky and difficult to conceptualize. The problem was to eliminate this lemma from proofs of Göodel's result. This was achieved only in the 1990s".

- Kleene, S. (1936 & 50) for Gödel's (& Rosser's) Theorem
- Robinson, A. (1963) for Tarski's Theorem
- Chaitin, G. (1970) for Gödel's Theorem
- Boolos, G. (1989) for Gödel's Theorem
- Caicedo, X. (1993) for Tarski's Theorem
- Jech, Th. (1994) for Gödel's 2nd Theorem
- Kotlarski, H. (1994 & 96 & 98) for Gödel's & Tarski's Theorems

The (Semantic) Diagonal Lemma (of Gödel & Carnap)

For every formula $\Psi(x)$ there exists a sentence η such that

$$\mathbb{N} \vDash \eta \leftrightarrow \Psi(\#\eta).$$

$\Longrightarrow$ Gödel's (Semantic) Incompleteness Theorem:

Proof.

Let $\mathtt{Pr}_T(x)$ define the set of provable sentences of the *recursively enumerable* theory T (i.e., $T \vdash \alpha \iff \mathbb{N} \vDash \mathtt{Pr}_T(\#\alpha)$); and let

$\mathbb{N} \vDash \gamma \leftrightarrow \neg\mathtt{Pr}_T(\#\gamma)$. *The Gödelian Sentence*

If $T \vdash \gamma$, then $\mathbb{N} \vDash \mathtt{Pr}_T(\#\gamma)$; also (by $\mathbb{N} \vDash \gamma$) $\mathbb{N} \vDash \neg\mathtt{Pr}_T(\#\gamma)$! ※

If $\mathbb{N} \vDash \neg\gamma$, then $\mathbb{N} \vDash \mathtt{Pr}_T(\#\gamma)$ and so $T \vdash \gamma$! ※

Thus, $\mathbb{N} \vDash \gamma$ and $T \nvdash \gamma$. Note that also $T \nvdash \neg\gamma$. □

The (Semantic) Diagonal Lemma (of Gödel & Carnap)

For every formula $\Psi(x)$ there exists a sentence η such that

$$\mathbb{N} \vDash \eta \leftrightarrow \Psi(\#\eta).$$

$\Longrightarrow$ Tarski's (Semantic) Undefinability Theorem:

Proof.
If $\Theta(x)$ defines the code set of *true* sentences $\{\#\eta \mid \mathbb{N} \vDash \eta\}$, then let
$\mathbb{N} \vDash \lambda \leftrightarrow \neg\Theta(\#\lambda)$. *The Liar's Paradox*
If $\mathbb{N} \vDash \lambda$, then $\mathbb{N} \vDash \Theta(\#\lambda)$ and also $\mathbb{N} \vDash \neg\Theta(\#\lambda)$! ⋇
If $\mathbb{N} \vDash \neg\lambda$, then $\mathbb{N} \vDash \Theta(\#\lambda)$ and so $\mathbb{N} \vDash \lambda$! ⋇ $\square$

The (Semantic) Diagonal Lemma (of Gödel & Carnap)

For every formula $\Psi(x)$ there exists a sentence η such that

$$\mathbb{N} \vDash \eta \leftrightarrow \Psi(\#\eta).$$

$\Longrightarrow$ Gödel's (General) Incompleteness Theorem:

Theorem

No sound and definable (deductively closed) theory is complete.

Proof.

Let $\Theta(x)$ define the sound deductively closed theory $T \subseteq \mathrm{Th}(\mathbb{N})$ (i.e., $T \vdash \alpha \iff \alpha \in T \iff \mathbb{N} \vDash \Theta(\#\alpha)$); and let $\mathbb{N} \vDash \gamma \leftrightarrow \neg\Theta(\#\gamma)$.

If $T \vdash \gamma$, then $\mathbb{N} \vDash \Theta(\#\gamma)$; also (by $\mathbb{N} \vDash \gamma$) $\mathbb{N} \vDash \neg\Theta(\#\gamma)$! ※

If $\mathbb{N} \vDash \neg\gamma$, then $\mathbb{N} \vDash \Theta(\#\gamma)$ and so $T \vdash \gamma$! ※

Thus, $\mathbb{N} \vDash \gamma$ and $T \nvdash \gamma$. Note that also $T \nvdash \neg\gamma$. □

Indeed, (Semantic) GÖDEL $\equiv$ (Semantic) TARSKI

Definition

Let $\mathscr{Th}_\Psi = \{\eta \mid \mathbb{N} \vDash \Psi(\#\eta)\}$ be the theory defined by $\Psi(x)$. ◀

theory = set of sentences

- (TARSKI) $\forall\Psi : \mathrm{Th}(\mathbb{N}) \neq \mathscr{Th}_\Psi$.
- (GÖDEL) $\forall T \subseteq \mathrm{Th}(\mathbb{N})$: $\exists\Psi(T = \mathscr{Th}_\Psi) \Longrightarrow T$ is <u>in</u>complete.

- (GÖDEL$\equiv$) $\forall T \subseteq \mathrm{Th}(\mathbb{N})$: T is complete $\Longrightarrow \forall\Psi(T \neq \mathscr{Th}_\Psi)$.

Fact

$\forall T \subseteq \mathrm{Th}(\mathbb{N})$: T *is complete*$^{\text{max. cons.}}$ $\Longrightarrow T = \mathrm{Th}(\mathbb{N})$. □

- (GÖDEL$\equiv$) $\forall^{\text{complete}} T \subseteq \mathrm{Th}(\mathbb{N})$: $\forall\Psi(T \neq \mathscr{Th}_\Psi)$.
  $\equiv \ \forall\Psi$: $\mathrm{Th}(\mathbb{N}) \neq \mathscr{Th}_\Psi$ ($\equiv$TARSKI).

Even, (Sem.) DIAGONAL LEMMA $\equiv$ (Sem.) TARSKI

We already saw (semantic) DIAGONAL LEMMA $\Longrightarrow$ (semantic) TARSKI

DIAGONAL LEMMA $\equiv \forall\,\Xi(x)\,\exists\eta\colon\ \mathbb{N}\vDash\eta\leftrightarrow\Xi(\#\eta)$

$\neg$DIAGONAL LEMMA $\equiv \exists\,\Xi(x)\,\forall\eta\colon\ \mathbb{N}\nvDash\eta\leftrightarrow\Xi(\#\eta)$

$\mathbb{N}\vDash\neg[\eta\leftrightarrow\Xi(\#\eta)]$

$\neg(p\leftrightarrow q)\equiv(p\leftrightarrow\neg q)$ $\qquad$ $\mathbb{N}\vDash\eta\leftrightarrow\neg\Xi(\#\eta)$

$\Psi(x)=\neg\Xi(x)$

$\neg$DIAGONAL LEMMA $\equiv \exists\,\Psi(x)\forall\eta\colon\ \mathbb{N}\vDash\eta\leftrightarrow\Psi(\#\eta)$

$\equiv \exists\,\Psi(x)\colon\ \mathrm{Th}(\mathbb{N})=\{\eta\mid\mathbb{N}\vDash\Psi(\#\eta)\}=\mathscr{T}\hbar_{\Psi}$

$\equiv \neg$TARSKI

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ON THE DIAGONAL LEMMA OF GÖDEL AND CARNAP*

SAEED SALEHI

Abstract.
A cornerstone of modern mathematical logic is the diagonal lemma of Gödel and Carnap. It is used in e.g. the classical proofs of the theorems of Gödel, Rosser and Tarski. From its first explication in 1934, just essentially one proof has appeared for the diagonal lemma in the literature; a proof that is so tricky and hard to relate that many authors have tried to avoid the lemma altogether. As a result, some so called diagonal-free proofs have been given for the above mentioned fundamental theorems of logic. In this paper, we provide new proofs for the semantic formulation of the diagonal lemma, and for a weak version of the syntactic formulation of it.

Beautiful Equivalences

Gödel (1931): The (Semantic) Incompleteness Theorem:

$\forall T \subseteq \mathrm{Th}(\mathbb{N})\colon \exists \Psi (T = \mathscr{T}\hbar_{\Psi}) \Longrightarrow T$ is <u>in</u>complete.

$\equiv$

Tarski (1933): The (Semantic) Undefinability Theorem:

$\forall \Psi\colon \mathrm{Th}(\mathbb{N}) \neq \mathscr{T}\hbar_{\Psi}$.

$\equiv$

Carnap (1934): The (Semantic) Diagonal Lemma:

$\forall\, \Xi(x)\, \exists \eta\colon \mathbb{N} \vDash \eta \leftrightarrow \Xi(\#\eta)$.

K. Lajevardi: where?
short proofs ...

A Case of Tarski (14.01.1901—26.10.1983) 120th birthday

Jan Mycielski, *A System of Axioms of Set Theory for the Rationalists*, **Notices of the AMS** 53:2(2006)206−213. www.ams.org/notices/200602/fea-mycielski.pdf

Tarski's Theorem (**Fundamenta Mathematicæ** 5:1(1924)147−154):
$\forall\kappa_{\textit{infinite cardinal}}(\kappa\cdot\kappa=\kappa)$ implies `The Axiom of Choice`.

The converse was known.

"[Tarski told me the following story. He tried to publish his theorem (stated above) in the *Comptes Rendus Acad. Sci. Paris* but Fréchet and Lebesgue refused to present it.
Fréchet wrote that an implication between two well known propositions is not a new result.
Lebesgue wrote that an implication between two false propositions is of no interest.
And Tarski said that after this misadventure he never tried to publish in the *Comptes Rendus*.]"

A Case of Alfred Tajtelbaum–Tarski (1924)today

ZF = Zermelo–Fraenkel Set Theory.

- Gödel, Kurt (1938); *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis*, **Proceedings of the National Academy of Sciences of the USA** 24(12):556–557.

  So, ZF + AC is consistent.

- Cohen, Paul J. (1963); *The Independence of the Continuum Hypothesis*, **Proceedings of the National Academy of Sciences of the USA** 50(6):1143–1148.

  So, ZF + ¬AC is consistent.

Tarski's Theorem: $\mathrm{ZF} \vdash \mathrm{AC} \longleftrightarrow \forall \kappa_{\textit{infinite cardinal}} (\kappa \cdot \kappa = \kappa)$.

Even More Beautiful Equivalences hot!

For a 1st order language $\mathcal{L}$ assume a *recursive & injective* coding $\#\colon \texttt{Sentences}(\mathcal{L}) \longrightarrow \texttt{Closed–Terms}(\mathcal{L})$.

Examples: $\mathcal{L} \supseteq \{1,+\}$ and so $\texttt{Closed–Terms}(\mathcal{L}) \supseteq \{\bar{n}\}_{n>0}$ where $\bar{n} = 1 + \cdots + 1$ (n-times).

For a structure $\mathcal{M}$ over $\mathcal{L}$, let $\mathscr{T}\hbar_{\Psi}^{\mathcal{M}} = \{\eta \mid \mathcal{M} \vDash \Psi(\#\eta)\}$; and

- $\mathbb{G}$ÖDEL$_{\mathcal{M}}$: $\forall T \subseteq \mathrm{Th}(\mathcal{M})\colon \exists \Psi (T = \mathscr{T}\hbar_{\Psi}^{\mathcal{M}}) \Longrightarrow T$ is <u>in</u>complete.
- $\mathbb{T}$ARSKI$_{\mathcal{M}}$: $\forall \Psi\colon \mathrm{Th}(\mathcal{M}) \neq \mathscr{T}\hbar_{\Psi}^{\mathcal{M}}$.
- $\mathbb{D}$IAGONAL$_{\mathcal{M}}$: $\forall\, \Xi(x)\, \exists \eta\colon \mathcal{M} \vDash \eta \leftrightarrow \Xi(\#\eta)$.

Theorem

For every $\langle \mathcal{L}, \#, \mathcal{M} \rangle$ we have $\mathbb{G}$ÖDEL$_{\mathcal{M}} \equiv \mathbb{T}$ARSKI$_{\mathcal{M}} \equiv \mathbb{D}$IAGONAL$_{\mathcal{M}}$.
For some $\langle \mathcal{L}, \#, \mathcal{M} \rangle$'s all three hold (such as PEANO Arithmetic) and for some $\langle \mathcal{L}, \#, \mathcal{M} \rangle$'s none holds (such as PRESBURGER Arithmetic). ■

Another Instance of Abstraction

LIAR's Paradox: $\boldsymbol{\lambda} \leftrightarrow \neg\boldsymbol{\lambda}$.

A (Propositional) Logical Tautology: $\neg(\boldsymbol{\lambda} \leftrightarrow \neg\boldsymbol{\lambda})$

p	$\neg p$	$p \to \neg p$	$\neg p \to p$	$p \leftrightarrow \neg p$	$\neg(p \leftrightarrow \neg p)$
t	f	f	t	f	t
f	t	t	f	f	t

RUSSELL's Paradox: $\mathcal{R} = \{x \mid x \notin x\}$.

$$\mathcal{R} \in \mathcal{R} \iff \mathcal{R} \in \{x \mid x \notin x\} \iff \mathcal{R} \notin \mathcal{R} \;\divideontimes$$

More Paradoxes …

So, $\nexists \mathcal{R}\colon \forall x(x \in \mathcal{R} \leftrightarrow x \notin x)$.

Or, $\mathrm{ZF} \vdash \neg\exists Y\, \forall x\, (x \in Y \leftrightarrow x \notin x)$.

Indeed, $\in$ is irrelevant!

BARBER's Paradox: $\nexists\, \mathrm{B}\, \forall x\, [\mathrm{shaves}(\mathrm{B}, x) \leftrightarrow \neg \mathrm{shaves}(x, x)]$.

since when?

Exercise 12, page 76: VAN DALEN, D.; *Logic and Structure*, Springer (5th ed. 2013).

$$\text{Predicate Logic} \vdash \neg\exists Y\, \forall x\, [\mathrm{r}(Y, x) \leftrightarrow \neg \mathrm{r}(x, x)].$$

Verber's Paradox:

someone says that she Verbs the ones and
only the ones who do **not** Verb themselves!

writing a biography…

Thank You!

Thanks to

The Participants . . . . For Listening · · ·

and

The Organizers — For Taking Care of Everything · · ·

Saeed Salehi.ir

A Birthday Present